How to acquire mental addition and subtraction skills: A review of text books available to support Initial Teacher Education students.

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Abstract

Trainee teachers in the England have to pass a mathematics skills test, which includes a section focusing on mental mathematics. In addition, all generalist primary school teachers are expected to teach mental mathematics strategies to children from 5 to 11 years of age, although many of these students have never been taught strategies themselves and therefore feel unprepared (Davis, 2009). What support is available to them to address this? This paper reviews the mental mathematics components of some of the key texts used by many Initial Teacher Education students, focusing on addition and subtraction strategies, and considers the extent to which these texts prepare students to teach mental strategies in school. I link these strategies and the representations displayed in the texts to research in this field and propose ways of enhancing the learning of these students which will ultimately impact on the future teaching of these key skills.

Key words: Mental Calculation Strategies; Primary ITE Students; Textbooks; Addition; Subtraction.

As a University tutor with eight years' experience of teaching both undergraduate and post graduate students training to be primary school teachers in the United Kingdom, I have observed that for many of these students there are a number of issues relating to the learning of the subject knowledge of mathematics which are not replicated in any other subject. In the UK very few primary teachers are specialists in mathematics, coming from all types of educational backgrounds. There is a minimum mathematics qualification requirement (Grade C at GCSE or equivalent) and often student teachers have received no mathematics education themselves since they achieved this qualification at the age of 16. As a result of this, often the first book that many students purchase, even before enrolment on a course, is a text book to support their knowledge, skills and understanding of the primary mathematics taught in school today. As I am particularly interested in mental calculation strategies I thought it would be important to look at some of the key professional texts which aim to support generalist teachers of primary mathematics and consider the extent to which they support students' learning of mental addition and subtraction skills. Although there are a number of different types of subject knowledge within mathematics, particularly pertaining to primary teachers (Goulding, Rowland and Barber, 2002; Shulman, 1986), for the purposes of this article I intend to consider subject content knowledge of students (as opposed to pedagogical knowledge).

The majority of students entering the teaching profession in England at the moment are those who have been educated after the publication of the influential Cockcroft Report (DES, 1982). This report highlighted the inability of a significant percentage of the UK adult population to apply the mathematics learned at school to their everyday lives, and who admitted to having a lack of confidence in the subject. Since the publication of this report there have been two radical changes in the teaching of mathematics in England, namely the introduction and subsequent revisions of the National Curriculum and a decade later the National Numeracy Strategy (DfEE, 1999) both of which aimed to raise standards in mathematics. However, this concern about the arithmetic ability of British society is not a new one, as can be noted by the comments of a senior member of Her Majesty's Inspectorate (HMI) who raised concerns in 1869 that teachers were keener to ensure children were passing examinations than they were to ensure they really understood the mathematics taught (Howson, cited in Brown 1999). One hundred and forty years later there are many who would argue that, as judgements about the quality of schools and teachers are based on test and examination results, the same is still true. This might also explain why, despite the changes in the school curriculum, I am still finding that there appears to be a lack of confidence in basic numeracy skills amongst student teachers who all achieved at least a grade C in their GCSE exams.

The particular skills that I am interested in here are those relating to mental calculation strategies, as the initial findings from previous research (Davis, 2009) indicate that many young adults do not possess a wide range of such strategies. In order to be recommended for Qualified Teacher Status (QTS) in England all student teachers (primary or secondary) have to pass skills tests in literacy, numeracy and ICT. The first part of the numeracy QTS skills test is designed to test their mental calculation skills, which appears to be the most challenging part of this test for most primary students. The idea behind this mental section was to ensure, 'trainees have an acceptable level of mental agility and recall to aid their interpretation, use and application of numerical information e.g. to be able to calculate mentally in everyday situations' (DfEE, 1998:7). The tenet underpinning the National Numeracy Project was that 'the ability to calculate mentally lies at the heart of numeracy' (Straker, 1999) but it would appear that the majority of current primary Initial Teacher Education (ITE) students in the UK, who just missed out on benefiting from this aspect of current teaching, have not acquired this ability during their education so far. It is therefore perhaps not surprising that they struggle with the mental section of the skills test and lack the confidence to teach mental maths in school.

There are many who advocate that the ability to select from a range of strategies is vital to show that a child is 'numerate' (Askew, 1999; DfEE, 1999; Evans, 2000). A numerate child, therefore, would be expected to choose a different strategy to calculate 201 – 198 than they might when calculating 76 – 11, although it is hoped that both of these would be calculated mentally rather than using a written method. If we are to accept that this area of mathematics is key to being numerate, and assuming that we would wish our primary teachers to be numerate, I thought it would be interesting to critically analyse three 'textbooks' which are commonly recommended to support the subject knowledge of student teachers, to see how they support the learning of mental strategies for addition and subtraction.

I began by considering how much of each book was devoted to this key area of mathematics. Haylock's popular book (2006), now with its fourth edition in the pipeline, has an entire chapter (twelve pages) devoted to mental strategies for addition and subtraction, beginning with an explanation of the commutative and associative laws. He includes a summary of some research carried out in Australia which is pertinent to mental calculations and the chapter is set out clearly with useful headings to guide the reader. At the end of the chapter there is guidance for further reading, a range of assessment questions, a glossary of any new terms introduced and a link to the relevant examples on the CD that is included with the book.

This level of commitment to mental mathematics was not so evident in the two further books that I initially chose to review. Suggate, Davis and Goulding (2001) embed their references to mental calculation strategies for addition and subtraction within a chapter which also includes counting, place value and written calculation methods. There are about two pages for mental addition and less than one and a half for mental subtraction strategies. At the end of the chapter, however, there is reference in the summary to the importance of developing 'robust mental methods' (2001:73) and on the same page there is also a recommendation that students discuss the statement, 'The traditional written algorithms are past their sell-by date'.

Reassuringly, since the publication of this book in 2001, Suggate, Davis and Goulding have acknowledged the increased importance of mental methods of calculation. The third edition of their text has now been published (2006) and in total almost five pages are now devoted to mental methods of addition and subtraction. A CD has also been included to enable students to practise choosing from a selection of methods, with representations including counters, tens and units blocks, number lines and number squares offered to model these processes.

The final book that I chose to review was the most surprising. Mooney et al (2009) set their references to mental calculation strategies within a very useful chapter on number. This chapter covers every aspect of number from the written methods currently used for each of the four operations (addition, subtraction, multiplication and division) to fractions, decimals, percentages, ratio and proportion. It also explains the laws of arithmetic, rational and irrational numbers and how to represent numbers in index and standard forms. These 34 pages form a reasonably comprehensive overview of our number system but there is less than half a page devoted to both mental addition and subtraction combined. Bearing in mind the current focus on teaching these skills it is surprising that this is not seen as a priority.

Having considered the amount of space devoted to mental calculation strategies in general, I moved on to looking at the strategies chosen by each author and how they used diagrams and other images to support the understanding of these examples.

Haylock (2006) begins his examples by explaining how important it is for children to be able to count forwards and backwards in units, tens and hundreds, from any given number, which he calls an 'essential prerequisite for effective mental calculation' (2006:45). He follows this with an explanation of how we can use multiples of ten as 'stepping stones' to aid both addition and subtraction. One example of this that he gives is when adding 28 to 57 he shows on an empty number line how we might initially add 20 to 57, making 77, then partition the 8 into '3 plus 5' in order to use 80 as a stepping stone. Diagrams support the understanding of this concept.

Haylock then moves on to describe a strategy where the most significant digits are added together first, which is in sharp contrast to the written methods that most of us use where we begin with the unit values. He is particularly clear in his explanation of this 'front end addition' where he partitions both numbers into hundreds, tens and units then adds together the hundreds, then the tens, then the units before recombining. His recording of his thinking process (2006:47), related to the mathematical laws, explains this well:

$$459 + 347 = (400 + 50 + 9) + (300 + 40 + 7)$$

$$= (400 + 300) + (50 + 40) + (9 + 7)$$

$$= 700 + 90 + 9 + 7$$

$$= 799 + 7 = 799 + 1 + 6 = 800 + 6 = 806$$

This final stage incorporates using 800 as a 'stepping stone'.

Mooney et al (2009) set out with clear intentions to cross-reference within and between chapters, so building on Wigley's view of a 'challenging model' of learning mathematics, namely that it is important to make connections within learning (1992). Unfortunately I could find no evidence of links being made in the brief sections on mental addition and subtraction strategies. Indeed, only one method is suggested for adding mentally, that of partitioning into hundreds, tens and units, then working 'left-to-right' (2009:12), which is the same method as Haylock's example shown above. Mooney et al's example of this (adding 234 and 325) is, however, shown in written format:

> 'Two hundred and three hundred gives five hundred. Thirty add twenty gives fifty, and four and five is nine. So it is five hundred and fifty nine.' (2009:12)

Another popular text, Suggate, Davis and Goulding, uses the same strategy as Haylock (above) for adding 45 and 38 together (2001:62)

40 + 30 = 70

70 + 8 = 78   (using the larger unit digit first)

78 + 5 = 83

In their most recent edition (Suggate, Davis and Goulding, 2006) their example of this method is set out differently and they now choose to add the units together before combining with the tens, rather than adding the larger unit digit first then the smaller one. The example this time is 35 + 28 (2006:68).

35 + 28 = (30 + 5) + (20 + 8)

= (30 + 20) + (5 + 8)

= 50 + 13

= 63

However, Suggate, Davis and Goulding also offer a different strategy for this same calculation, a method whereby only the second number is partitioned (2001:62):

As 38 = 30 + 5 + 3

45 + 38   can be thought of as

45 + 30 = 75 then

75 + 5 = 80 then

80 +3

83

Interestingly, not only have they chosen a different example to demonstrate this in their most recent edition, they have also chosen not to partition the 8 into 5 and 3 this time (2006:68).

35 + 28 = 35 + (20 + 8)

= (35 + 20) + 8

= 55 + 8

= 63

Beishuizen (1993) describes work he carried out in the Netherlands, comparing these two strategies but just with numbers up to one hundred. He refers to the strategy which all three texts included, as ‘1010’, i.e. separating the tens and adding them, then the units. This method was found to be the easiest for children to understand to begin with, probably because it uses known number facts. In Haylock’s (2006) example 400 and 300 can easily be added as it uses the number facts of 4 + 3 and so forth. However, the second strategy that Suggate, Davis and Goulding (2001; 2006) suggested, which Beishuizen calls ‘N10’, i.e. keeping the first number as a whole then adding multiples of 10 to it, was found to be ultimately the ‘more efficient in terms of mental response times’ (1993:318). This confirmed the earlier findings of Wolters et al (cited in Beishuizen 1993). A useful ‘N10’ example based on the calculation that Haylock used to exemplify the ‘1010’ approach would therefore be:

459 + 347 = 459 + (300 + 40 + 7)

= (459 + 300) + (40 + 7)

= (759 + 40) + 7

= 799 + 7 = 799 + 1 + 6 = 800 + 6 = 806

A further strategy for addition, the compensatory method, was exemplified by two of these authors. Haylock chose to demonstrate this method by using an empty number line (figure 1) to calculate 673 + 99 (2006:48)

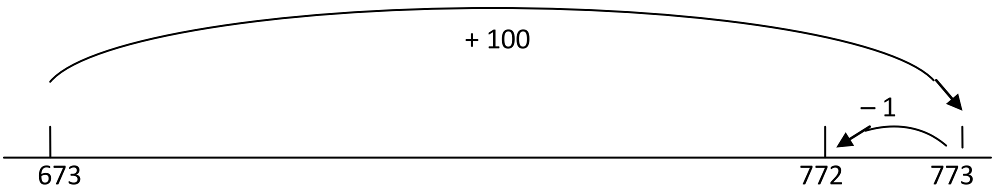

Figure 1. Using an empty number line for adding 673 + 99

Suggate, Davis and Goulding (2001) added 45 and 38 together using this method, but they used this more traditional model to exemplify this process before showing the same calculation on a model of an empty number line (2001:62).

45 + 38 = 40 + 5 + 40 – 2

= 80 + 5 – 2

= 80 + 3

= 83

A particular strength of Suggate, Davis and Goulding's textbook (2001) is that it makes it clear why they have chosen particular diagrams to support these strategies. These diagrams, drawn from research, can be used as mental images to support mental calculations, for example the use of a number line is linked to work in the Netherlands and the use of the number square is based on Lacey's work (cited in Suggate, Davis and Goulding 2001:63). The final image to support mental addition is that of a number square which is extended indefinitely in each direction (*figure 2)*. Whilst I accept that this is an image I am unfamiliar with, my initial thoughts about it are that I think the extensions to right and left could be confusing for students and children, as the same number appears in different places. It is certainly not an image I have seen being taught in primary schools and I am sceptical as to its value with young children. It does, however, combine the image of the number square with that of the number line (2001:63).

-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9
-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
8	9	10	11	12	13	14	15	16	17	18	19	20	21
18	19	20	21	22	23	24	25	26	27	28	29	30	31
28	29	30	31	32	33	34	35	36	37	38	39	40	41
38	39	40	41	42	43	44	45	46	47	48	49	50	51
48	49	50	51	52	53	54	55	56	57	58	59	60	61
58	59	60	61	62	63	64	65	66	67	68	69	70	71
68	69	70	71	72	73	74	75	76	77	78	79	80	81
78	79	80	81	82	83	84	85	86	87	88	89	90	91
88	89	90	91	92	93	94	95	96	97	98	99	100	101
98	99	100	101	102	103	104	105	106	107	108	109	110	111
108	109	110	111	112	113	114	115	116	117	118	119	120	121

Figure 2. Number Square

In their most recent edition (Suggate, Davis and Goulding, 2006), many of the examples and images are now missing from the text book and instead a CD has been included which they hope will ‘contribute to the proper understanding of the methods concerned’ (2006:1). On this CD there are representations displayed for a selection of examples and although some of these representations are useful for some of the calculations, there are some obvious omissions, for example there is no option to start with the larger number rather than the first number in the calculation 16 + 27. I am also concerned that there is no recommendation as to which representation might be most useful for each particular calculation. How are students to learn to select the most effective representations to use when teaching strategies to children? Also, whilst I strongly commend the use of an empty number line as an image to support mental addition and subtraction, I am unconvinced that an empty number square is of much use (see figure 3 for a representation of adding 27 to 16 by ‘overjumping’ shown on the CD).

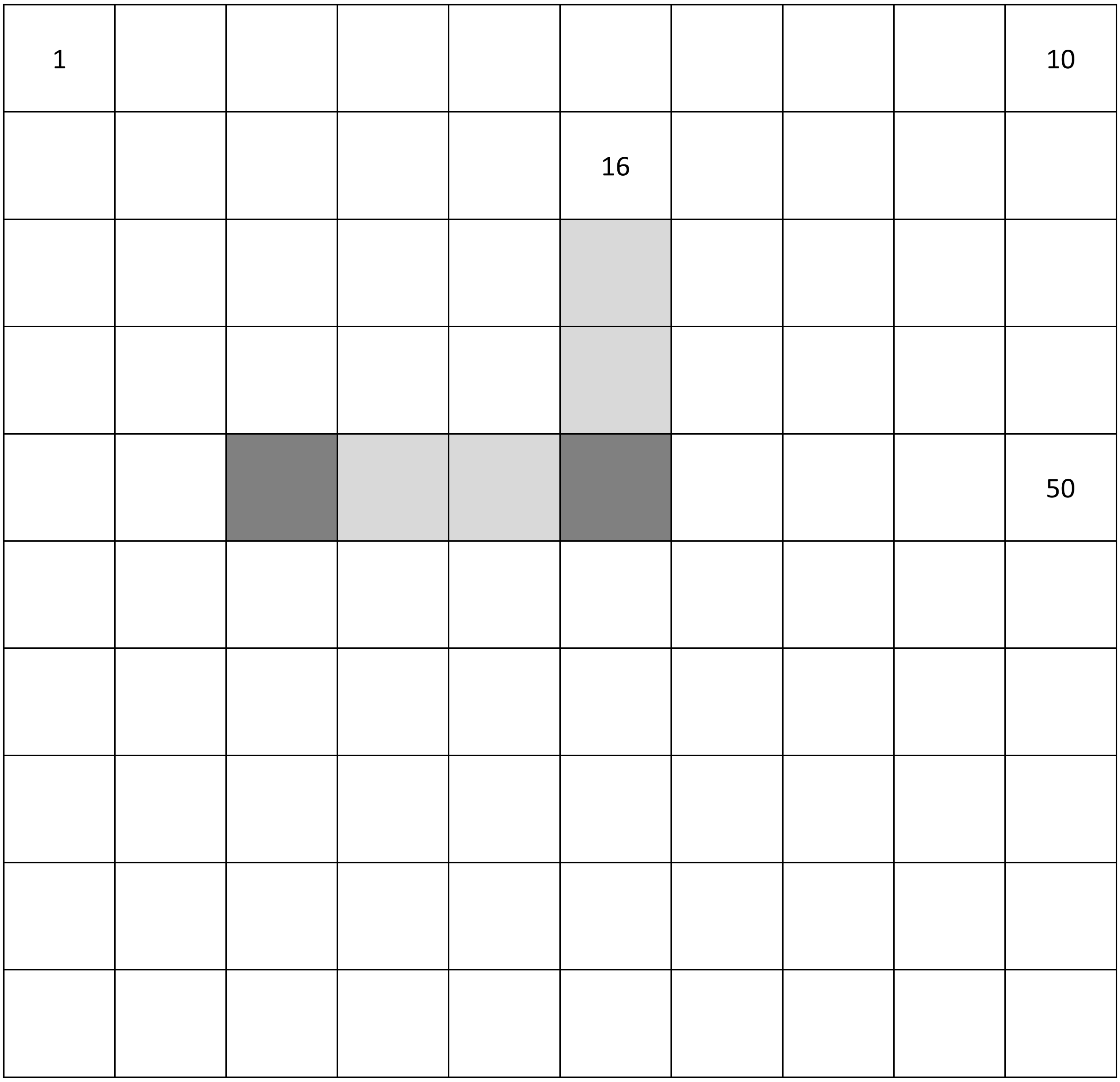

Figure 3. Using an empty number square to calculate 16 + 27 by 'overjumping'.

Other strategies Haylock describes for mental addition are using multiples of 5 and using doubles, both of which I would agree are useful strategies for particular calculations. When considering doubles, Haylock's first example is 48 + 46, which he then records as:

46 + 46 = 92, so 48 + 46 = 92 + 2 = 94 (2006:48)

Whilst this is, of course, an appropriate strategy, I would also want to draw students' attention to the fact that 48 + 46 = 47 + 47. Although I found no evidence of this in any of these books I agree with Sugarman, amongst others, who believes 'transforming to retain equivalence' (cited in Thompson 1999:4) is a key mental strategy. I would also argue that this idea of equivalence is so embedded in many areas of mathematics it is useful for children to become familiar with it as early as possible. I would argue that it is accessible to children by the time they are able to mentally calculate 48 + 46, although Thompson (1999) is not convinced that young children are able to understand the idea of equivalence, as in his research he only found one child out of 350 using this strategy. If both student teachers and experienced teachers understood this idea of equivalence and taught it effectively, maybe more children would use it as a mental strategy.

Haylock generally discusses subtraction strategies alongside addition, using similar models to demonstrate the strategies of using stepping stones to calculate 542 – 275 and compensation to calculate 83 – 28 (Figure 4).

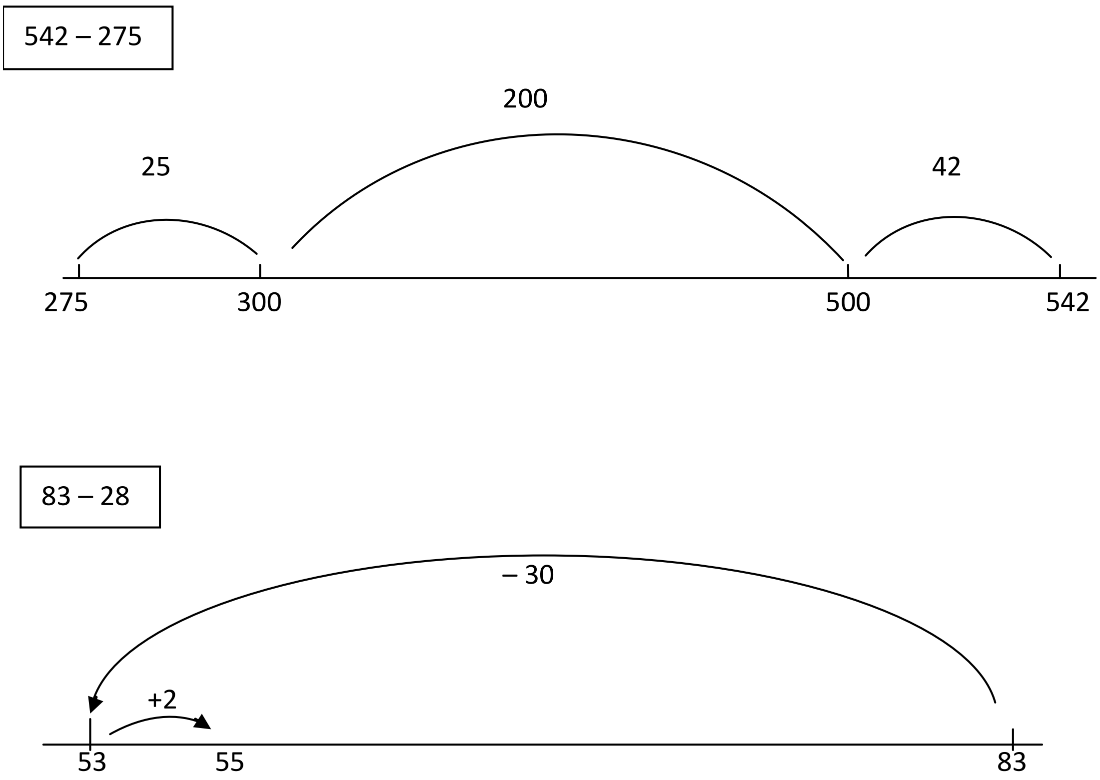

Figure 4. Demonstrating ‘stepping stones’ and ‘compensation’ on an empty number line

Haylock’s final suggestion for both addition and subtraction is that of using ‘friendly numbers’, although his example is one of subtraction. His suggestion is that we alter one of the numbers to something that links more closely to the other, then compensate after completing the calculation. His example of 742 – 146, therefore is as follows (2006:51):

Change the 146 to 142: 742 – 142 = 600

Now compensate: 742 – 146 = 600 – 4 = 596

Or

Change the 742 to 746: 746 – 146 = 600

Now compensate: 742 – 146 = 600 – 4 = 596

Suggate, Davis and Goulding use the same method (which they call chunking) to calculate 63 - 35 (2001:68; 2006:74):

35 = 33 – 2

So 63 – 35

= 63 – 33 – 2

= 30 – 2

= 28

Although Haylock provides useful strategies for subtraction as well as addition, we must take particular care if teaching one of his suggested approaches to mental subtraction, in order that it does not lead to a misconception. Following his front-end addition example (above), he states that we can quite often use the front-end approach to start us off in mental subtraction, ‘for example, for 645 – 239, we would immediately deal with the hundreds (600 – 200 = 400) leaving us simply to think about 45 – 39’ (2006:44). With student teachers (and ultimately the children they teach) we need to draw attention to the fact that this is only possible when the tens digit in the larger number is greater than the tens digit in the smaller one, so, for example this might not be the most efficient approach for 645 – 253.

In terms of subtraction strategies, Suggate, Davis and Goulding (2001; 2006) make clear links to their previous discussion of place value and the idea of subtraction as taking away. This then leads to a discussion of the benefits of counting on or ‘complimentary addition’, and they use a number line (figure 5) to demonstrate how to mentally calculate 63 – 35 (2001:68; 2006:72).

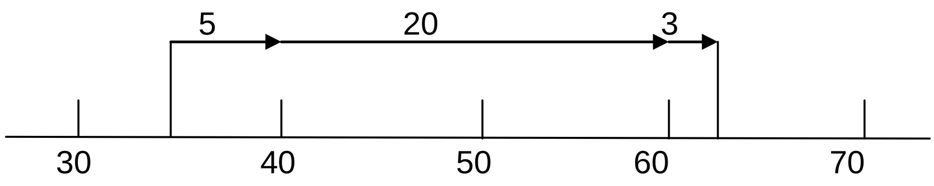

Figure 5. Using an empty number line to demonstrate subtraction by 'complimentary addition'

It is, however, surprising to find just two strategies for subtraction in Suggate, Davis and Goulding (2001), namely this complementary addition and the chunking method demonstrated earlier. This is particularly disappointing when one considers their comment at the end of this section that 'the range of mental methods for subtraction is probably greater than those for addition' (2001:68). The more recent edition (2006) also discusses partitioning both numbers but, quite rightly, explains the problems that may arise from this method, for example if we try to subtract 35 from 63 by partitioning both numbers we finish up with (60 – 30) + (3 – 5). There are more examples on the CD, raising identical issues surrounding representations to those I described for addition.

Finally, only one mental subtraction method, that of complimentary addition, is shown in Mooney et al (2006:10), and that merely as a precursor to a written format.

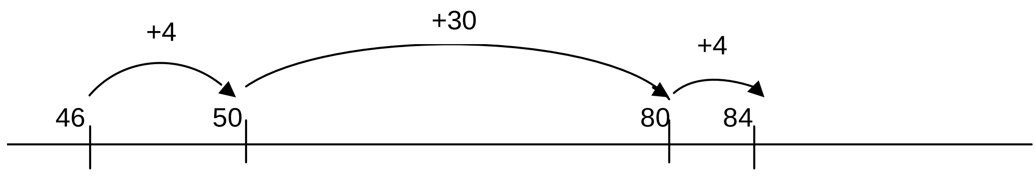

Figure 6. Using an empty number line to demonstrate subtraction by 'complimentary addition'

This example, in contrast to their addition example shown earlier, does at least have a diagram of a horizontal number line (figure 6), demonstrating how to count on in steps from 46 to 84, but interestingly, the most recent advice from the Primary National Strategy is that a vertical number line should be used, not a horizontal one, as this allows for jottings to be placed in a column alongside the line (DfES, 2005). None of these books used this recommended orientation.

Overall, Haylock's approach of instruction and explanation, followed by examples then self-assessment questions, ending with summary bullet points is a particularly useful format, based on the constructivist theory of learning.

Conclusion

It is vital that student teachers understand that there are a range of strategies available for mental calculation and that they have some idea of the most widely used in English primary schools today. This survey of current textbooks indicates that although the strategies given within these popular student books are by no means comprehensive, between them they have selected most of the key ones to introduce to primary teachers. However it is likely that student teachers will still need support in learning a range of mental calculation strategies, so I will now offer two suggestions for providing this support, with rationales for each suggestion.

Askew (1999) argues strongly that he found that the more highly skilled teachers were those who were basing their teaching on a 'connectionist' approach to learning. His belief that children need to make connections between different aspects, symbols and methods of mathematics supports the view that to be able to calculate efficiently and effectively mentally, children (and, by implication, student teachers) need to be able to choose 'the best' strategy for any given calculation. By providing opportunities for them to make connections between, for example, using doubles and certain addition calculations we are not only enabling them to become more numerate, but we are also developing their general learning skills. This links with Treffers' learning principles that mathematics is a constructive activity, not a body of knowledge that can just be transmitted, and that there needs to be an 'intertwining of the various learning strands within mathematics teaching' (cited in Treffers and Beishuizen 1999:36).

Secondly, much has been written about the benefit of discussion and interactions with others when learning mathematics (Hughes, Deforges and Mitchell, 2000; Carter, 2005; Mercer and Sams, 2006) and this seems to me to be particularly beneficial when learning mental calculation strategies. Providing students with the opportunities to discuss their variety of methods would not only enable them to select the most appropriate method but also help them to extend their own repertoire of strategies. Turner (2009) agrees that this interaction within their own community of practice enables teachers to develop their 'mathematical content knowledge' much more strongly. Of these three key texts only Haylock referred to this aspect of learning, and this was advice to encourage pupils to discuss methods, rather than an encouragement of the students to do this for themselves.

I began this article by suggesting that mental calculation skills are often perceived as being an indication of a person's ability in numeracy. It appears that the textbooks available to students training to become generalist teachers in our primary schools (and therefore with responsibilities to enable the next generation to become more numerate) provide students with some suggestions of strategies. In order for these strategies to become part of a repertoire of strategies from which the most effective and efficient methods can be chosen, we also need to provide them with the opportunity to discuss their strategies in a supportive environment so that they can make links between different aspects of their understanding of number. This skill of being able to select an efficient and effective method is at the heart of using mental rather than written strategies, so how do students know what to look for first? Goulding, Rowland and Barker (2002) reported that poor subject matter knowledge was associated with poor planning and teaching of primary mathematics, so it is crucial that we strengthen this knowledge for all of our student teachers.

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